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LETTER TO THE EDITOR

Pressure and stress tensor expressions in the fluid mechanical formulation of the Bose condensate equations

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Abstract. It is shown that a momentum equation

$$\frac{\partial}{\partial t}(\rho u_i)+\frac{\partial}{\partial x_j}(\rho u_i u_j+p\delta_{ij}-\sigma_{ij}') = 0,$$

identical in form to that of the familiar Navier–Stokes fluid, can be derived for the fluid condensate of a weakly interacting Bose gas. For the condensate the pressure is given by the simple barotropic relation $p = \rho^2/4$, while the anisotropic stress tensor σ_{ij}' depends only on the density and density gradients and is given by

$$\sigma_{ij}' = \left(\frac{\rho\partial^2\rho}{\partial x_i\,\partial x_j} - \frac{\partial\rho}{\partial x_i}\frac{\partial\rho}{\partial x_j}\right)(4\rho)^{-1}.$$

During the past decade the condensate of a gas of weakly interacting Bose particles has frequently been studied as a model for liquid helium near absolute zero. Gross (1963) has shown that when all the particles comprising the Bose gas are 'condensed' into the lowest state the system can be described by a macroscopic wavefunction $\psi(\boldsymbol{x}, t)$ which satisfies the nonlinear field equation

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2M}\nabla^2\psi+V_0\psi|\psi|^2, \tag{1}$$

and also a normalization condition that

$$\int_V|\psi|^2\mathrm{d}V = N. \tag{2}$$

Here M is the mass of the boson, N is the total number of particles in the condensate, which occupies a volume V, and in deriving equation (1) the repulsive potential between bosons $\boldsymbol{x}$ and $\boldsymbol{y}$ has been represented by a short range potential of delta function type $V_0\delta(\boldsymbol{x}-\boldsymbol{y})$.

By writing the complex wavefunction $\psi(\boldsymbol{x}, t)$ in the form

$$\psi(\boldsymbol{x}, t) = R(\boldsymbol{x}, t)\exp\left(\frac{\mathrm{i}S(\boldsymbol{x}, t)}{\hbar}\right), \tag{3}$$

and taking the real and imaginary parts of (1), Gross has further shown that the field equations assume a hydrodynamic form in which R^2 and S are representative of the density and velocity potential respectively. Actually $\rho = MR^2$ and $\phi = -(E_V t+S)/M$ (where E_V is a constant) are a better choice for the density and potential and by suitably choosing the length scale, unit of mass and unit of time Roberts and Grant (1971) have reduced the condensate equations to

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \boldsymbol{u}) = 0, \tag{4}$$

$$u^2 + \rho - \frac{\nabla^2 \rho^{1/2}}{\rho^{1/2}} - 2\frac{\partial \phi}{\partial t} = 1, \tag{5}$$

where

$$\boldsymbol{u} = -\nabla\phi, \tag{6}$$

which have a distinct fluid mechanical appearance. The normalization condition (2) becomes

$$\int_V \rho \mathrm{d}V = 1. \tag{7}$$

Equation (4) is the equation of continuity and equation (5) the 'Bernoulli' equation for the fluid. Several exact solutions of the equations (4) to (7) now exist. Roberts and Grant (1971), Grant (1971) and Grant and Roberts (1974) have used the equations to analyse the structures and oscillation spectra of vortex rings and lines and also the structure and effective masses of charged and uncharged impurities, while Tsuzuki (1971) has shown the existence of interesting nonlinear wave solutions. In some respects we have a fluid which is amenable to more satisfactory theoretical treatment than the Navier–Stokes fluid.

From a mathematical viewpoint it is the occurrence of the term $-\nabla^2\rho^{1/2}/\rho^{1/2}$ in the Bernoulli equation which often makes such solutions extractable, allowing equation (5) to be solved for the density. So far this term has been loosely referred to as a 'quantum pressure' term. In the hope of generating further interest in the condensate equations among physicists and fluid mechanicists alike it is the purpose of the present letter to point out that the fluid mechanical formulation of the condensate equations can be carried further (Grant 1972), yielding expressions for the pressure in the condensate and indicating the existence and form of an anisotropic stress tensor, and thus allowing striking comparisons from a physical viewpoint with the familiar Navier–Stokes fluid.

To see this we may first note that (unlike the Navier–Stokes equations) equations (4) and (5) can be obtained from a variational principle of the form

$$\delta \int\int L \mathrm{d}V \mathrm{d}t = 0, \tag{8}$$

subject to the constraint (7), if the lagrangian L is given by

$$L = \frac{\rho}{2}\left(\frac{\partial \phi}{\partial x_i}\right)^2 + \frac{1}{8\rho}\left(\frac{\partial \rho}{\partial x_i}\right)^2 + \frac{\rho^2}{4} - \rho\frac{\partial \phi}{\partial t}. \tag{9}$$

This enables us to write down immediately some conservation laws for the system. Using the invariance of L with respect to an arbitrary change in x_i we obtain

$$\begin{aligned} \frac{\partial}{\partial t}\left(-\frac{\rho\partial\phi}{\partial x_i}\right) + \frac{\partial}{\partial x_j}\left(\frac{\partial \phi}{\partial x_i}\left(\frac{\rho\partial\phi}{\partial x_j}\right) + \frac{1}{4\rho}\frac{\partial\rho}{\partial x_i}\frac{\partial\rho}{\partial x_j}\right) \\ - \frac{\partial}{\partial x_i}\left[\frac{\rho}{2}\left(\frac{\partial\phi}{\partial x_j}\right)^2 + \frac{1}{8\rho}\left(\frac{\partial\rho}{\partial x_j}\right)^2 + \frac{\rho^2}{4} - \frac{\rho\partial\phi}{\partial t} - \frac{\rho}{2}\right] = 0, \end{aligned} \tag{10}$$

which upon eliminating $\partial\phi/\partial t$ using (5) reduces to

$$\frac{\partial}{\partial t}(\rho u_i)+\frac{\partial}{\partial x_j}\Pi_{ij}=0, \tag{11}$$

where the momentum flux density tensor Π_{ij} is defined by

$$\Pi_{ij}=p\delta_{ij}+\rho u_i u_j-\sigma_{ij}', \tag{12}$$

and the pressure p and anisotropic quantum stress tensor σ_{ij}' are given by the relations

$$p=\frac{\rho^2}{4}, \qquad \sigma_{ij}'=\frac{1}{4}\left(\frac{\partial^2\rho}{\partial x_i\,\partial x_j}-\frac{1}{\rho}\frac{\partial\rho}{\partial x_i}\frac{\partial\rho}{\partial x_j}\right). \tag{13}$$

Equation (11) is identical to the momentum equation for the viscous Navier–Stokes fluid (see for example Landau and Lifshitz 1959, p 47). As is evident, however, the stress components in the condensate depend only upon the density and density gradients in the fluid, in contrast to those of the viscous stress tensor for the Navier–Stokes fluid which depend only upon velocity gradients; the pressure in the condensate is given by a simple baratropic relation, an interesting but unusual feature in fluid mechanics.

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